

The optimal control related to Riemannian manifolds and the viscosity solutions to H-J-B equations

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Abstract

This paper is concerned with the Dynamic Programming Principle (DPP in short) with SDEs on Riemannian manifolds. Moreover, through the DPP, we conclude that the cost function is the unique viscosity solution to the related PDEs on manifolds.

Keywords: Dynamic programming principle; Riemannian manifold; Viscosity solution.

1 Introduction

El Karoui, Peng and Quenez [3] gave the formulation of recursive utilities and their properties from the BSDE point of view. As we know, the recursive optimal control problem is represented as a kind of optimal control problem whose cost functional is described by the solution of BSDE. In 1992, Peng [5] got the Bellman's dynamic programming principle for this kind of problem and proved that the value function is a viscosity solution of one kind of quasi-linear second-order partial differential equation (PDE in short) which is the well-known as Hamilton-Jacobi-Bellman (H-J-B in short) equation. Later in 1997, he virtually generalized these results to a much more general situation, under Markvian and even Non-Markvian framework ([6], Chapter 2).

But sometimes, in financial market, as the solution to a SDE with control, the wealth process of the investor may be constrained, for example, it should be nonnegative. In particular, for some special need, it may be a process in some curving spaces. So it is natural to consider the following question: if the SDE in stochastic recursive optimal control problems is defined on Riemannian manifolds, then will we still have the similar results as what we have mentioned in R^n ? The objective of this paper is to study this problem.

Let $(W(t), t \geq 0)$ be a d -dimensional standard Brownian motion on some complete probability space (Ω, \mathcal{F}, P) . We denote by $(\mathcal{F}_t)_{t \geq 0}$ the natural filtration generated by W and augmented by the P -null sets of \mathcal{F} .

Let U be a compact subset of R^{d+1} . We call a function $f : \Omega \times [t, T] \rightarrow U$ an admissible control if it's an adapted stochastic process. We denote by $\mathcal{U}_{t,T}$ the set of all admissible controls.

Assume that M is a compact Riemannian manifold without boundary. Now we can consider the following controlled stochastic differential equation on M in a fixed time interval $[t, T]$:

$$\begin{cases} dX_s^{t,\zeta;v} = v_0(s)V_0(s, X_s^{t,\zeta;v})ds + \sum_{\alpha=1}^d V_\alpha(s, X_s^{t,\zeta;v}) \circ v_\alpha(s)dW_s^\alpha, \\ X_t^{t,\zeta;v} = \zeta \in M, \end{cases} \quad (1.1)$$

where ζ is \mathcal{F}_t -measurable, $v. = v(\cdot) := (v_0(\cdot), v_1(\cdot), \dots, v_d(\cdot)) \in \mathcal{U}_{t,T}$, and V_0, V_1, \dots, V_d are $d+1$ deterministic one-parameter smooth vector fields on M .

Since M is compact and without boundary, according to [4], there exists a unique M -valued continuous process which solves equation (1.1). Moreover, this solution does not explode.

Let us consider functions $f : [0, T] \times M \times R \times R^{1 \times d} \times U \rightarrow R$ and $\Phi : M \rightarrow R$ which satisfy:

(A1).there exists a constant $K \geq 0$, s.t., we have: $\forall t, \forall (x, y, z, v)$ and (x', y', z', v') ,

$$|\Phi(x) - \Phi(x')| + |f(t, x, y, z, v) - f(t, x', y', z', v')| \leq K(|y - y'| + |z - z'| + d(x, x') + |v - v'|),$$

(A2).there exists a constant $K_0 \geq 0$, s.t., $\forall (t, x, v)$, $|f(t, x, 0, 0, v)| \leq K_0$,

where $d(\cdot, \cdot)$ denotes the Riemannian distance function on M .

By the above assumptions, according to [6], there exists a unique solution $(Y., Z.) \in \mathcal{M}(t, T; R \times R^{1 \times d})$ to the following BSDE:

$$\begin{cases} -dY_s^{t,\zeta;v} = f(s, X_s^{t,\zeta;v}, Y_s^{t,\zeta;v}, Z_s^{t,\zeta;v}, v_s) - Z_s^{t,\zeta;v}dW_s, s \in [t, T], \\ Y_T^{t,\zeta;v} = \Phi(X_T^{t,\zeta;v}), \end{cases}$$

where $\mathcal{M}(0, T; R^n)$ denotes the Hilbert space of adapted stochastic processes $f : \Omega \times [0, T] \rightarrow R^n$ such that

$$\|f\| = (E \int_0^T |f(t)|^2 dt)^{\frac{1}{2}} < \infty.$$

When $\zeta = x \in M$ is deterministic, We define

$$J(t, x; v(\cdot)) := Y_s^{t, x; v} |_{s=t}.$$

This is the so-called cost function. And then we can define a value function of the optimal control problem as follows:

$$u(t, x) := \text{essinf}_{v(\cdot) \in \mathcal{U}_{t,T}} J(t, x; v(\cdot)).$$

Our purpose is to get the general Dynamic Programming Principle of the value function $u(t, x)$.

2 Dynamic Programming Principle

If we define

$$\mathcal{U}_{t,T}^t := \{v(\cdot) \in \mathcal{U}_{t,T} : v(\cdot) \text{ is } \mathcal{F}_s^t \text{-adapted}\},$$

where $\mathcal{F}_s^t := \sigma\{W_r - W_t, t \leq r \leq s\}$.

By Proposition 5.1 in [6], there exist $\{v^i(\cdot)\}_{i=1}^\infty, v^i(\cdot) \in \mathcal{U}_{t,T}^t$, such that $u(t, x) = \lim_{i \rightarrow \infty} J(t, x; v^i(\cdot))$ and $u(t, x)$ is a deterministic function, i.e.,

$$u(t, x) := \text{essinf}_{v(\cdot) \in \mathcal{U}_{t,T}} J(t, x; v(\cdot)) = \inf_{v(\cdot) \in \mathcal{U}_{t,T}^t} J(t, x; v(\cdot)).$$

Since our SDE is defined on Riemannian manifolds, $\forall(\zeta, v(\cdot))$ and $(\zeta', v'(\cdot))$, $d^2(X_s^{t, \zeta; v}, X_s^{t, \zeta'; v'})$ is not necessarily twice differentiable. So the good estimate about the continuous dependence of $X_s^{t, \zeta; v}$ w.r.t to (ζ, v) does not hold and nor does $Y_s^{t, \zeta; v} |_{s=t}$. They're unfavourable factors for our dynamic programming principle. So we turn to the embedding mapping.

By the Whitney's theorem, there exists an embedding mapping Ψ such that, $\Psi : M \rightarrow \Psi(M) \subset R^n$ for some $n \in N$. Set $\Psi(X_s^{t, \zeta; v}) = \tilde{X}_s^{t, \tilde{\zeta}; v}$, where $\tilde{\zeta} = \Psi(\zeta)$. Then $\tilde{X}_t^{t, \tilde{\zeta}; v}$ satisfies the following SDE on $\Psi(M)$:

$$\begin{cases} d\tilde{X}_s^{t, \tilde{\zeta}; v} = v_0(s) \tilde{V}_0(s, \tilde{X}_s^{t, \tilde{\zeta}; v}) ds + \sum_{\alpha=1}^d \tilde{V}_\alpha(s, \tilde{X}_s^{t, \tilde{\zeta}; v}) \circ v_\alpha(s) dW_s^\alpha, \\ \tilde{X}_t^{t, \tilde{\zeta}; v} = \tilde{\zeta} \in \Psi(M), \end{cases} \quad (2.1)$$

where $\tilde{V}_\alpha = \Psi_* V_\alpha$, $\alpha = 0, 1, \dots, d$ and Ψ_* is the tangent mapping. And we can extend each \tilde{V}_α to smooth vector field defined on R^n with compact support. We denote the extensions still by \tilde{V}_α .

So we have the following SDE in R^n whose initial point is in $\Psi(M)$:

$$\begin{cases} d\tilde{X}_s^{t,\tilde{\zeta};v} = v_0(s)\tilde{V}_0(s, \tilde{X}_s^{t,\tilde{\zeta};v})ds + \sum_{\alpha=1}^d v_\alpha(s)\tilde{V}_\alpha(s, \tilde{X}_s^{t,\tilde{\zeta};v})dW_s^\alpha + \frac{1}{2} \sum_{\alpha=1}^d v_\alpha^2(s)\nabla_{\tilde{V}_\alpha} \tilde{V}_\alpha(s, \tilde{X}_s^{t,\tilde{\zeta};v})ds, \\ \tilde{X}_t^{t,\tilde{\zeta};v} = \tilde{\zeta} \in \Psi(M), \end{cases} \quad (2.2)$$

where ∇ is the connection of R^n . According to [4], SDE (2.2) has the same unique solution with SDE (2.1), i.e., although although SDE (2.2) is defined in R^n , as long as its initial point is in $\Psi(M)$, it won't leave $\Psi(M)$.

Since U is compact and each \tilde{V}_i is a smooth vector field in R^n with compact support, the coefficients of SDE (2.2) are bounded and Lipschitz continuous w.r.t. x and v . By [6], we have the following estimate:

$$E^{\mathcal{F}_t}[\sup_{s \in [t, T]} |\tilde{X}_s^{t,\tilde{\zeta};v} - \tilde{X}_s^{t,\tilde{\zeta}';v'}|^2] \leq C(|\tilde{\zeta} - \tilde{\zeta}'|^2 + E^{\mathcal{F}_t} \int_t^T |v(s) - v'(s)|^2 ds), \quad (2.3)$$

where $C > 0$ is a constant which only depends on the Lipschitz constant of the coefficients of SDE (2.2). Here and in the sequel, the constant C appearing in each estimate won't be necessarily the same one.

Lemma 2.1. $Y_t^{t,x;v}$ is continuous w.r.t $(x, v(\cdot))$ and it is uniformly continuous in x , uniformly in $(t, v(\cdot))$.

Proof: Using Itô's formula to $e^{\beta(s-t)}|Y_s^{t,x;v} - Y_s^{t,x';v'}|^2$ for some positive constant β , we have

$$\begin{aligned} & |Y_t^{t,x;v} - Y_t^{t,x';v'}|^2 + E^{\mathcal{F}_t} \int_t^T e^{\beta(s-t)} [\beta |Y_s^{t,x;v} - Y_s^{t,x';v'}|^2 + |Z_s^{t,x;v} - Z_s^{t,x';v'}|^2] ds \\ &= e^{\beta(T-t)} E^{\mathcal{F}_t} |\Phi(X_T^{t,x;v}) - \Phi(X_T^{t,x';v'})|^2 + E^{\mathcal{F}_t} \int_t^T e^{\beta(s-t)} 2(Y_s^{t,x;v} - Y_s^{t,x';v'}) * \\ & \quad (f(s, X_s^{t,x;v}, Y_s^{t,x;v}, Z_s^{t,x;v}, v_s) - f(s, X_s^{t,x';v'}, Y_s^{t,x';v'}, Z_s^{t,x';v'}, v'_s)) ds \\ &\leq K e^{\beta(T-t)} E^{\mathcal{F}_t} d^2(X_T^{t,x;v}, X_T^{t,x';v'}) + \frac{1}{2} E^{\mathcal{F}_t} \int_t^T e^{\beta(s-t)} |Z_s^{t,x;v} - Z_s^{t,x';v'}|^2 ds \\ & \quad + E^{\mathcal{F}_t} \int_t^T e^{\beta(s-t)} d^2(X_s^{t,x;v}, X_s^{t,x';v'}) ds + E^{\mathcal{F}_t} \int_t^T e^{\beta(s-t)} |v_s - v'_s|^2 ds \\ & \quad + E^{\mathcal{F}_t} \int_t^T e^{\beta(s-t)} (2K + 2K^2 + K^2 + K^2) |Y_s^{t,x;v} - Y_s^{t,x';v'}|^2 ds. \end{aligned}$$

If we choose $\beta = 2K + 4K^2 + 1$, we have

$$\begin{aligned}
& |Y_t^{t,x;v} - Y_t^{t,x';v'}|^2 + E^{\mathcal{F}_t} \int_t^T e^{\beta(s-t)} [|Y_s^{t,x;v} - Y_s^{t,x';v'}|^2 + \frac{1}{2} |Z_s^{t,x;v} - Z_s^{t,x';v'}|^2] ds \\
& \leq C \{ E^{\mathcal{F}_t} [\sup_{s \in [t,T]} d^2(X_s^{t,x;v}, X_s^{t,x';v'})] \\
& \quad + E^{\mathcal{F}_t} \int_t^T \sup_{s \in [t,T]} d^2(X_s^{t,x;v}, X_s^{t,x';v'}) ds + E^{\mathcal{F}_t} \int_t^T |v_s - v'_s|^2 ds \}.
\end{aligned} \tag{2.4}$$

Since M is compact, $\Psi : M \rightarrow \Psi(M)$ and $\Psi^{-1} : \Psi(M) \rightarrow M$ are both uniformly continuous mappings. So with (2.3), when $(x', v') \rightarrow (x, v)$, we have

$$E^{\mathcal{F}_t} [\sup_{s \in [t,T]} d^2(X_s^{t,x;v}, X_s^{t,x';v'})] \rightarrow 0.$$

Moreover, when $x' \rightarrow x$,

$$\sup_{v(\cdot) \in \mathcal{U}_{t,T}} E^{\mathcal{F}_t} [\sup_{s \in [t,T]} d^2(X_s^{t,x;v}, X_s^{t,x';v})] \leq E^{\sup_{v(\cdot) \in \mathcal{U}_{t,T}} \mathcal{F}_t} [\sup_{s \in [t,T]} d^2(X_s^{t,x;v}, X_s^{t,x';v})] \rightarrow 0. \tag{2.5}$$

Through the theorem of control convergence, we have that $Y_t^{t,x;v}$ is continuous w.r.t $(x, v(\cdot))$.

What's more, $\forall \varepsilon > 0$, we choose $\delta_0 = \frac{\varepsilon}{C(1+T)}$. By (2.5), for this δ_0 , there exists $\delta > 0$, such that, when $d(x, x') < \delta$ (here δ doesn't depend on x or x'),

$$\sup_{v(\cdot) \in \mathcal{U}_{t,T}} E^{\mathcal{F}_t} [\sup_{s \in [t,T]} d^2(X_s^{t,x;v}, X_s^{t,x';v})] < \delta_0.$$

Combining (2.4), we have

$$\begin{aligned}
& |Y_t^{t,x;v} - Y_t^{t,x';v}|^2 + E^{\mathcal{F}_t} \int_t^T e^{\beta(s-t)} [|Y_s^{t,x;v} - Y_s^{t,x';v}|^2 + \frac{1}{2} |Z_s^{t,x;v} - Z_s^{t,x';v}|^2] ds \\
& < C\delta_0 + CT\delta_0 = \varepsilon.
\end{aligned}$$

So we have finished the proof. \square

And we can get some properties of $u(t, x)$:

Lemma 2.2. $u(t, x)$ is bounded and uniformly continuous in x , uniformly in t .

Proof: Applying Itô's formula to $e^{\beta_1(s-t)} |Y_s^{t,x;v}|^2$ ($\beta_1 = 2K + 2K^2 + 2$), with the same method of above, we have

$$\begin{aligned}
& |Y_t^{t,x;v}|^2 + E^{\mathcal{F}_t} \int_t^T \beta_1 e^{\beta_1(s-t)} |Y_s^{t,x;v}|^2 ds + E^{\mathcal{F}_t} \int_t^T e^{\beta_1(s-t)} |Z_s^{t,x;v}|^2 ds \\
& \leq e^{\beta_1(T-t)} E^{\mathcal{F}_t} |\Phi(X_T^{t,x;v})| \\
& \quad + E^{\mathcal{F}_t} \int_t^T e^{\beta_1(s-t)} 2 |Y_s^{t,x;v}| (K |Y_s^{t,x;v}| + K |Z_s^{t,x;v}| + |f(s, X_s^{t,x;v}, 0, 0, v_s)|) ds.
\end{aligned}$$

Since $\Phi(\cdot)$ is continuous and M is compact, $\Phi(x)$ is bounded. This with A2, we have that there exists some constant C independent of $(t, x, v(\cdot))$, such that,

$$|Y_t^{t,x;v}|^2 + E^{\mathcal{F}_t} \int_t^T e^{\beta_1(s-t)} |Y_s^{t,x;v}|^2 ds + \frac{1}{2} E^{\mathcal{F}_t} \int_t^T e^{\beta_1(s-t)} |Z_s^{t,x;v}|^2 ds \leq C. \quad (2.6)$$

So $u(t, x) \leq C$.

By the definition of $u(t, x)$, we know that for any $\varepsilon > 0$, there exist $v(\cdot), v'(\cdot) \in \mathcal{U}_{t,T}^t$, such that,

$$Y_t^{t,x;v} - \varepsilon \leq u(t, x), \quad Y_t^{t,x';v'} - \varepsilon \leq u(t, x').$$

So we have

$$Y_t^{t,x;v} - \varepsilon \leq u(t, x) \leq Y_t^{t,x;v'}, \quad Y_t^{t,x';v'} - \varepsilon \leq u(t, x') \leq Y_t^{t,x';v}.$$

And that yields

$$Y_t^{t,x;v} - Y_t^{t,x';v'} - \varepsilon \leq u(t, x) - u(t, x') \leq Y_t^{t,x;v'} - Y_t^{t,x';v'} + \varepsilon.$$

Since $Y_t^{t,x;v}$ is continuous in x uniformly in $(t, v(\cdot))$, we've got our conclusion. \square

If we replace the variable x in $u(t, x)$ by a r.v. ζ which is \mathcal{F}_t -measurable, we have:

Lemma 2.3. For any fixed $t \in [0, T]$ and ζ which is \mathcal{F}_t -measurable, we have:

(i) $\forall v(\cdot) \in \mathcal{U}_{t,T}, u(t, \zeta) \leq Y_t^{t,\zeta;v}$,

(ii) $\forall \varepsilon > 0$, there exists a $v(\cdot) \in \mathcal{U}_{t,T}$ such that $u(t, \zeta) \geq Y_t^{t,\zeta;v} - \varepsilon$.

Proof: We have known that u is continuous in x and $Y_t^{t,\zeta;v}$ is continuous in $(\zeta, v(\cdot))$. Recall that the collection of processes $(v(s))_{s \in [t,T]}$ with

$$\{v(s) = \sum_{i=1}^N I_{A_i} v^i(s) : \{A_i\}_{i=1}^N \text{ is a } \mathcal{F}_t\text{-partition of } \Omega, v^i(\cdot) \in \mathcal{U}_{t,T} \text{ is } \mathcal{F}_s^t\text{-adapted}\}$$

is dense in $\mathcal{U}_{t,T}$. So for (i), we need only to discuss special ζ and $v(\cdot)$ as follows:

$$\zeta = \sum_{i=1}^N I_{A_i} x_i, v(\cdot) = \sum_{i=1}^N I_{A_i} v^i(\cdot),$$

where A_i and $v^i(\cdot)$ are described as above, and $x_i \in M, i = 1, \dots, N$. Then we can use the same method with Theorem 4.7 in [6] to get

$$Y_t^{t,\zeta;v} = \sum_{i=1}^N I_{A_i} Y_t^{t,x_i;v^i} \geq \sum_{i=1}^N I_{A_i} u(t, x_i) = u(t, \sum_{i=1}^N I_{A_i} x_i) = u(t, \zeta).$$

For (ii), we can use the same technique. Considering that M is compact, for any $n \in N$, there exist a collection of data $\{U_i, \varphi_i\}_{i=1}^{N_n}$ such that $\dim(U_i) < \frac{1}{2^n}$, where $\dim(U_i) := \sup_{x,y \in M} d(x,y)$. For any ζ which is \mathcal{F}_t -measurable, choose any fixed $y_i \in U_i$ and set $\eta_n = \sum_{i=1}^{N_n} y_i I_{\{\omega: \zeta(\omega) \in U_i\}}$. Then we have

$$d(\eta_n, \zeta) < \frac{1}{2^n}, P - a.s.$$

By Lemma 2.1. and 2.2., for any $\varepsilon > 0$, there exists $\delta > 0$, such that, when $d(x, x') < \delta$,

$$u(t, x) - u(t, x') \geq -\frac{\varepsilon}{3}, \quad Y_t^{t,x;v} - Y_t^{t,x';v} \geq -\frac{\varepsilon}{3}, \quad \text{for any } (t, v(\cdot)) \in [0, T] \times \mathcal{U}_{t,T}.$$

For this δ , through the discussion above, there exists $\eta = \sum_{i=1}^{N_\delta} x_i I_{A_i}$ where $x_i \in M$ and $\{A_i\}_{i=1}^{N_\delta}$ is a \mathcal{F}_t -partition of Ω , such that $d(\eta, \zeta) < \delta, P - a.s..$ So we have, for any $(t, v(\cdot)) \in [0, T] \times \mathcal{U}_{t,T}$,

$$u(t, \zeta) \geq u(t, \eta) - \frac{\varepsilon}{3}, \quad Y_t^{t,\eta;v} - Y_t^{t,\zeta;v} \geq -\frac{\varepsilon}{3}, P - a.s.. \quad (2.7)$$

On the other hand, there exists $v^i(\cdot) \in \mathcal{U}_{t,T}$, such that,

$$u(t, x_i) \geq Y_t^{t,x_i;v^i} - \frac{\varepsilon}{3}, i = 1, 2, \dots, N_\delta.$$

So

$$u(t, \eta) = \sum_{i=1}^{N_\delta} I_{A_i} u(t, x_i) \geq \sum_{i=1}^{N_\delta} I_{A_i} Y_t^{t,x_i;v^i} - \frac{\varepsilon}{3} = Y_t^{t,\eta;v} - \frac{\varepsilon}{3},$$

where $v(\cdot) = \sum_{i=1}^{N_\delta} I_{A_i} v^i(\cdot)$. This with (2.7), we have

$$u(t, \zeta) \geq Y_t^{t,\zeta;v} - \varepsilon.$$

□

Before stating the generalized Dynamic Programming Principle, let us recall the following basic estimate of the solutions to BSDEs which will be used often in the sequel (see Theorem 2.3. in [6]):

Lemma 2.4. Consider the following two BSDEs:

$$Y_t^1 = \xi^1 + \int_t^T [g(s, Y_s^1, Z_s^1) + \varphi_s^1] ds - \int_t^T Z_s^1 dW_s, \quad (a)$$

$$Y_t^2 = \xi^2 + \int_t^T [g(s, Y_s^2, Z_s^2) + \varphi_s^2] ds - \int_t^T Z_s^2 dW_s, \quad (b)$$

where $\xi^1, \xi^2 \in L^2(\Omega, \mathcal{F}_T, P; R^m)$, $\varphi^1, \varphi^2 \in \mathcal{M}(t, T; R^m)$, and $g : \Omega \times [0, T] \times R^m \times R^{m \times d} \rightarrow R^m$ satisfies: $\forall (y, z) \in R^m \times R^{m \times d}$, $g(\cdot, y, z)$ is a \mathcal{F}_t -adapted process valued in R^m and

$$\int_0^T |g(\cdot, 0, 0)| ds \in L^2(\Omega, \mathcal{F}_T, P; R^m),$$

$$|g(t, y, z) - g(t, y', z')| \leq C_L(|y - y'| + |z - z'|).$$

Then the difference between the solutions to BSDE (a) and (b) satisfies:

$$\begin{aligned} & |Y_t^1 - Y_t^2|^2 + \frac{1}{2} E^{\mathcal{F}_t} \int_t^T [|Y_s^1 - Y_s^2|^2 + |Z_s^1 - Z_s^2|^2] e^{\beta_0(s-t)} ds \\ & \leq E^{\mathcal{F}_t} |\xi^1 - \xi^2|^2 e^{\beta_0(T-t)} + E^{\mathcal{F}_t} \int_t^T |\varphi_s^1 - \varphi_s^2|^2 e^{\beta_0(s-t)} ds, \end{aligned}$$

where $\beta_0 = 16(1 + C_L^2)$.

Now let's consider the so-called backward semigroup (see [6]): $\forall (t, x) \in [0, T] \times M, 0 \leq \delta \leq T - t, \eta \in L^2(\Omega, \mathcal{F}_{t+\delta}, P; R)$, we set

$$G_{t,t+\delta}^{t,x;v.}[\eta] := Y_t,$$

where $(Y_s, Z_s)_{t \leq s \leq t+\delta}$ is the unique solution to the following BSDE:

$$\begin{cases} -dY_s = f(s, X_s^{t,x;v.}, Y_s, Z_s, v_s) - Z_s dW_s, s \in [t, t+\delta], \\ Y_{t+\delta} = \eta. \end{cases}$$

So obviously,

$$G_{t,T}^{t,x;v.}[\Phi(X_T^{t,x;v.})] = G_{t,t+\delta}^{t,x;v.}[Y_{t+\delta}^{t,x;v.}].$$

The follows is the generalized Dynamic Programming Principle (DPP in short):

Theorem 2.5. $\forall (t, x) \in [0, T] \times M, \forall \delta \in [0, T - t]$, we have

$$\begin{aligned} u(t, x) &= \text{essinf}_{v(\cdot) \in \mathcal{U}_{t,t+\delta}} G_{t,t+\delta}^{t,x;v.}[u(t+\delta, X_{t+\delta}^{t,x;v.})] \\ &= \inf_{v(\cdot) \in \mathcal{U}_{t,t+\delta}^t} G_{t,t+\delta}^{t,x;v.}[u(t+\delta, X_{t+\delta}^{t,x;v.})]. \end{aligned} \quad (2.8)$$

Proof: We will only prove the first equality. By the definition of $u(t, x)$, we have

$$\begin{aligned} u(t, x) &= \text{essinf}_{v(\cdot) \in \mathcal{U}_{t,T}} G_{t,T}^{t,x;v.}[\Phi(X_T^{t,x;v.})] \\ &= \text{essinf}_{v(\cdot) \in \mathcal{U}_{t,T}} G_{t,t+\delta}^{t,x;v.}[Y_{t+\delta}^{t+\delta, X_{t+\delta}^{t,x;v.}; v.}]. \end{aligned}$$

So by Lemma 2.3. and the comparison theorem of BSDEs, we get

$$u(t, x) \geq \text{essinf}_{v(\cdot) \in \mathcal{U}_{t,t+\delta}} G_{t,t+\delta}^{t,x;v.}[u(t+\delta, X_{t+\delta}^{t,x;v.})].$$

On the other hand, for any $\varepsilon > 0$, there exists an admissible control $\bar{v}(\cdot) \in \mathcal{U}_{t+\delta, T}$, such that,

$$u(t+\delta, X_{t+\delta}^{t,x;v.}) \geq Y_{t+\delta}^{t+\delta, X_{t+\delta}^{t,x;v.}; \bar{v}.} - \varepsilon.$$

Also by the comparison theorem of BSDEs and Lemma 2.4., we have

$$\begin{aligned}
u(t, x) &= \text{essinf}_{v(\cdot) \in \mathcal{U}_{t,T}} G_{t,t+\delta}^{t,x;v} [Y_{t+\delta}^{t+\delta, X_{t+\delta}^{t,x;v}; v}] \\
&\leq \text{essinf}_{v(\cdot) \in \mathcal{U}_{t,t+\delta}} G_{t,t+\delta}^{t,x;v} [Y_{t+\delta}^{t+\delta, X_{t+\delta}^{t,x;v}; \bar{v}}] \\
&\leq \text{essinf}_{v(\cdot) \in \mathcal{U}_{t,t+\delta}} G_{t,t+\delta}^{t,x;v} [u(t+\delta, X_{t+\delta}^{t,x;v}) + \varepsilon] \\
&\leq \text{essinf}_{v(\cdot) \in \mathcal{U}_{t,t+\delta}} G_{t,t+\delta}^{t,x;v} [u(t+\delta, X_{t+\delta}^{t,x;v})] + C\varepsilon.
\end{aligned}$$

Since ε is arbitrary, (2.8) holds true. □

In Lemma 2.2., we know that $u(t, x)$ is uniformly continuous in x , uniformly in t . Now with the DPP, we can also get that $u(t, x)$ is continuous in t .

Proposition 2.6. The value function $u(t, x)$ is continuous in $t, t \in [0, T]$.

Proof: $\forall (t, x) \in [0, T] \times M, \delta \in [0, T-t]$, by the DPP, we have: $\forall \varepsilon > 0$, there exists an admissible control $\bar{v}(\cdot) \in \mathcal{U}_{t,T}^t$, such that,

$$G_{t,t+\delta}^{t,x;\bar{v}} [u(t+\delta, X_{t+\delta}^{t,x;\bar{v}})] \geq u(t, x) \geq G_{t,t+\delta}^{t,x;\bar{v}} [u(t+\delta, X_{t+\delta}^{t,x;\bar{v}})] - \varepsilon. \quad (2.9)$$

So

$$u(t, x) - u(t+\delta, x) \leq G_{t,t+\delta}^{t,x;\bar{v}} [u(t+\delta, X_{t+\delta}^{t,x;\bar{v}})] - u(t+\delta, x) = I_\delta^1 + I_\delta^2, \quad (2.10)$$

where

$$\begin{aligned}
I_\delta^1 &= I_\delta^1(\bar{v}(\cdot)) = G_{t,t+\delta}^{t,x;\bar{v}} [u(t+\delta, X_{t+\delta}^{t,x;\bar{v}})] - G_{t,t+\delta}^{t,x;\bar{v}} [u(t+\delta, x)], \\
I_\delta^2 &= I_\delta^2(\bar{v}(\cdot)) = G_{t,t+\delta}^{t,x;\bar{v}} [u(t+\delta, x)] - u(t+\delta, x).
\end{aligned}$$

Now let us evaluate I_δ^1 and I_δ^2 . Still by Lemma 2.4., we have

$$|I_\delta^1| \leq [C_0 E |u(t+\delta, X_{t+\delta}^{t,x;\bar{v}}) - u(t+\delta, x)|^2]^{\frac{1}{2}}. \quad (2.11)$$

We can use the similar method as in (3.6) to get that

$$E \left[\sup_{v \in \mathcal{U}_{t,T}^t} d^2(X_{t+\delta}^{t,x;v}, x) \right] \rightarrow 0, \quad \text{when } \delta \rightarrow 0.$$

Thus for all $\varepsilon_0 > 0$,

$$\lim_{\delta \rightarrow 0} \sup_{v \in \mathcal{U}_{t,T}^t} P\{d^2(X_{t+\delta}^{t,x;v}, x) > \varepsilon_0\} = 0.$$

By the continuity of u w.r.t x , we have

$$\lim_{\delta \rightarrow 0} \sup_{v \in \mathcal{U}_{t,T}^t} \sup_{s \in [t, t+\delta]} P\{|u(s, X_{t+\delta}^{t,x;v}) - u(s, x)|^2 > \varepsilon_0\} = 0.$$

Recall that $u(t, x)$ is bounded, so

$$\lim_{\delta \rightarrow 0} \sup_{v \in \mathcal{U}_{t,T}^t} \sup_{s \in [t, t+\delta]} E|u(s, X_{t+\delta}^{t,x;\bar{v}}) - u(s, x)|^2 = 0.$$

Through (2.11), we know that

$$\lim_{\delta \rightarrow 0} \left[\sup_{v \in \mathcal{U}_{t,T}^t} |I_\delta^1(v(\cdot))| \right] = 0.$$

For I_δ^2 , we have

$$\begin{aligned} I_\delta^2 &= E[u(t+\delta, x) + \int_t^{t+\delta} f(s, X_s^{t,x;\bar{v}}, Y_s^{t,x;\bar{v}}, Z_s^{t,x;\bar{v}}, \bar{v}_s) ds - \int_t^{t+\delta} Z_s^{t,x;\bar{v}} dW_s] - u(t+\delta, x) \\ &= E[\int_t^{t+\delta} f(s, X_s^{t,x;\bar{v}}, Y_s^{t,x;\bar{v}}, Z_s^{t,x;\bar{v}}, \bar{v}_s) ds]. \end{aligned}$$

From the assumptions (A1) and (A2),

$$\begin{aligned} |I_\delta^2| &\leq E \int_t^{t+\delta} K(|Y_s^{t,x;\bar{v}}| + |Z_s^{t,x;\bar{v}}|) ds + K_0 \delta \\ &\leq C(\int_t^{t+\delta} E(|Y_s^{t,x;\bar{v}}|^2 + |Z_s^{t,x;\bar{v}}|^2) e^{\beta_1(s-t)} ds)^{\frac{1}{2}} + K_0 \delta, \end{aligned}$$

where β_1 is a constant defined in Lemma 2.2. Note (2.6) in Lemma 2.2., we have

$$\lim_{\delta \rightarrow 0} \left[\sup_{v \in \mathcal{U}_{t,T}^t} |I_\delta^2(v(\cdot))| \right] = 0.$$

From (2.10), we can conclude that

$$\limsup_{\delta \rightarrow 0} [u(t, x) - u(t+\delta, x)] \leq 0. \quad (2.12)$$

Consider the right inequality of (2.9), we have

$$u(t, x) - u(t+\delta, x) \geq G_{t,t+\delta}^{t,x;\bar{v}}[u(t+\delta, X_{t+\delta}^{t,x;\bar{v}})] - u(t+\delta, x) - \varepsilon = I_\delta^1 + I_\delta^2 - \varepsilon.$$

Thus

$$\liminf_{\delta \rightarrow 0} [u(t, x) - u(t+\delta, x)] \geq \liminf_{\delta \rightarrow 0} \left[- \sup_{v \in \mathcal{U}_{t,T}^t} |I_\delta^1(v(\cdot))| - \sup_{v \in \mathcal{U}_{t,T}^t} |I_\delta^2(v(\cdot))| \right] - \varepsilon.$$

So for any $\varepsilon > 0$,

$$\liminf_{\delta \rightarrow 0} [u(t, x) - u(t+\delta, x)] \geq -\varepsilon.$$

That is

$$\liminf_{\delta \rightarrow 0} [u(t, x) - u(t+\delta, x)] \geq 0.$$

This with (2.12), we get that $\lim_{\delta \rightarrow 0} [u(t, x) - u(t+\delta, x)] = 0$.

□

Remark 2.7. We can conclude from this proposition and Lemma 2.2. that $u(t, x)$ is continuous in $(t, x) \in [0, T] \times M$.

3 The viscosity solution to the generalized Hamilton-Jacobi-Bellman equation on Rimeannian manifolds

As it is well known to all, the value function $u(t, x)$ is usually a viscosity solution to some Hamilton-Jacobi-Bellman(H-J-B in short) equation. Our generalized H-J-B equation is:

$$\begin{cases} \partial_t u(t, x) + \inf_{v \in U} \{ (v_0 V_0 u)(t, x) + \frac{1}{2} \sum_{\alpha=1}^d (v_\alpha^2 V_\alpha V_\alpha u)(t, x) + f(t, x, u, \{v_\alpha V_\alpha u\}_{\alpha=1}^d, v) \} = 0, \\ u(T, x) = \Phi(x), \end{cases} \quad (3.1)$$

where $v = (v_0, v_1, \dots, v_d) \in U$ and we denote by $\{v_\alpha V_\alpha u\}_{\alpha=1}^d = (v_1 V_1 u, \dots, v_d V_d u)$ an element in $R^{1 \times d}$. This is a fully nonlinear second order parabolic PDEs on Riemannian manifolds.

The theory of viscosity solutions to PDEs in Euclidean space was introduced by M. G. Crandall and P. L. Lions in the 1980's (see [2]). Until recently, D. Azagra, J. Ferrera and B. Sanz [1] gave a work about Dirichlet problem on a complete Riemannian manifold with some restrictions on curvature. X. Zhu [7] studied parabolic PDEs on Riemannian manifolds.

Definition 3.1. We say $u \in C([0, T] \times M)$ is a viscosity supersolution (subsoulution) of (3.1), if $u(T, x) \geq \Phi(x)$ ($\leq \Phi(x)$), and for all $\varphi \in C^{1,2}([0, T] \times M)$, at each minimum (maximum) point (t, x) of $u - \varphi$ and $u(t, x) = \varphi(t, x)$, the following inequality holds:

$$\partial_t \varphi(t, x) + \inf_{v \in U} \{ (v_0 V_0 \varphi)(t, x) + \frac{1}{2} \sum_{\alpha=1}^d (v_\alpha^2 V_\alpha V_\alpha \varphi)(t, x) + f(t, x, u, \{v_\alpha V_\alpha \varphi\}_{\alpha=1}^d, v) \} \leq 0 (\geq 0).$$

$u(t, x)$ is said to be a viscosity solution of (3.1) if it is both a viscosity supersolution and a subsoulution.

We have proved that $u(t, x)$ is continuous in $(t, x) \in [0, T] \times M$. So we are ready to present that $u(t, x)$ is a viscosity solution of (3.1).

Theorem 3.2. Under the assumptions (A1) and (A2), the value function $u(t, x)$ is a viscosity solution to H-J-B equation (3.1).

To prove this theorem, we need the following three lemmas. We set

$$F(s, x, y, z, v) = \partial_t \varphi(s, x) + (v_0 V_0 \varphi)(s, x) + \frac{1}{2} \sum_{\alpha=1}^d (v_\alpha^2 V_\alpha V_\alpha \varphi)(s, x) + f(s, x, y + \varphi(s, x), z + \{v_\alpha V_\alpha \varphi\}_{\alpha=1}^d, v),$$

and consider the a BSDE defined on $[t, t + \delta]$:

$$\begin{cases} -dY_s^{1;v} = F(s, X_s^{t,x;v}, Y_s^{1;v}, Z_s^{1;v}, v_s) - Z_s^{1;v} dW_s, \\ Y_{t+\delta}^{1;v} = 0, \end{cases} \quad (3.2)$$

Then let us consider following first lemma:

Lemma 3.3. $\forall s \in [t, t + \delta]$, we have

$$Y_s^{1;v} = G_{s,t+\delta}^{t,x,v}[\varphi(t + \delta, X_{t+\delta}^{t,x;v})] - \varphi(s, X_s^{t,x;v}), a.s..$$

Proof: Recall that $G_{s,t+\delta}^{t,x,v}[\varphi(t + \delta, X_{t+\delta}^{t,x;v})]$ is defined through the solution to the following BSDE:

$$\begin{cases} -dY_s^v = f(s, X_s^{t,x;v}, Y_s^v, Z_s^v, v_s) - Z_s^v dW_s, s \in [t, t + \delta], \\ Y_{t+\delta}^v = \varphi(t + \delta, X_{t+\delta}^{t,x;v}). \end{cases} \quad (3.3)$$

That is

$$G_{s,t+\delta}^{t,x,v}[\varphi(t + \delta, X_{t+\delta}^{t,x;v})] = Y_s^v, s \in [t, t + \delta].$$

So what we need to do is proving that $Y_s^{1;v} + \varphi(s, X_s^{t,x;v}), s \in [t, t + \delta]$ is also a solution to (3.3).

Applying Itô's formula to $Y_s^{1;v} + \varphi(s, X_s^{t,x;v})$:

$$\begin{aligned} & -d(Y_s^{1;v} + \varphi(s, X_s^{t,x;v})) \\ = & F(s, X_s^{t,x;v}, Y_s^{1;v}, Z_s^{1;v}, v_s)ds - Z_s^{1;v}dW_s \\ & - [\partial_t \varphi(s, X_s^{t,x;v}) + (v_0 V_0 \varphi)(s, X_s^{t,x;v}) + \frac{1}{2} \sum_{\alpha=1}^d (v_\alpha^2 V_\alpha V_\alpha \varphi)(s, X_s^{t,x;v})]ds \\ & - \{(v_\alpha V_\alpha \varphi)(s, X_s^{t,x;v})\}_{\alpha=1}^d dW_s \\ = & f(s, X_s^{t,x;v}, Y_s^{1;v} + \varphi(s, X_s^{t,x;v}), Z_s^{1;v} + \{(v_\alpha V_\alpha \varphi)(s, X_s^{t,x;v})\}_{\alpha=1}^d) \\ & - (Z_s^{1;v} + \{(v_\alpha V_\alpha \varphi)(s, X_s^{t,x;v})\}_{\alpha=1}^d) dW_s. \end{aligned}$$

Moreover,

$$(Y_s^{1;v} + \varphi(s, X_s^{t,x;v}))_{s=t+\delta} = \varphi(t + \delta, X_{t+\delta}^{t,x;v}).$$

So $Y_s^{1;v} + \varphi(s, X_s^{t,x;v}), s \in [t, t + \delta]$ is in fact a solution to (3.3). By the uniqueness of the solution to (3.2), we've finished the proof. □

Consider the following BSDE which is easier than (3.2):

$$\begin{cases} -dY_s^{2;v} = F(s, x, Y_s^{2;v}, Z_s^{2;v}, v_s) - Z_s^{2;v} dW_s, s \in [t, t + \delta], \\ Y_{t+\delta}^{2;v} = 0. \end{cases} \quad (3.4)$$

The following lemma shows that, when δ is small enough, the difference between BSDE(3.2) and (3.4) $|Y_t^{1;v} - Y_t^{2;v}|$ can be ignored.

Lemma 3.4. We have the following estimate:

$$|Y_t^{1;v} - Y_t^{2;v}| \leq C\delta\rho_1(\delta), \quad (3.5)$$

where, $\rho_1(\delta) \downarrow 0$, when $\delta \downarrow 0$, and $\rho_1(\cdot)$ does not depend on the control $v(\cdot) \in \mathcal{U}_{t,t+\delta}^t$.

Proof: If we set

$$\eta^\delta := \sup_{s \in [t, t+\delta]} d(X_s^{t,x;v}, x).$$

With the similar method as in (3.6), We have

$$E[\eta^\delta] \downarrow 0, \text{ when } \delta \downarrow 0. \quad (3.6)$$

We can use the estimate in Lemma 2.4. to BSDE(3.2) and (3.4) with $\xi^1 = \xi^2 = 0$,

$$\begin{aligned} g(s, y, z) &= F(s, X_s^{t,x;v}, y, z, v_s), \\ \varphi_s^1 &= 0, \varphi_s^2 = F(s, x, Y_s^{2;v}, Z_s^{2;v}, v_s) - F(s, X_s^{t,x;v}, Y_s^{2;v}, Z_s^{2;v}, v_s). \end{aligned}$$

From the definition of the function F , we can see that $g(s, y, z)$ satisfies Lipschitz conditions w.r.t. (y, z) . And there exists a constant C and a function $\rho(\cdot)$ with

$$\rho(\delta) \downarrow 0, \text{ when } \delta \downarrow 0$$

which do not depend on $v(\cdot)$, such that,

$$|\varphi_s^2| \leq C\rho(d(X_s^{t,x;v}, x)),$$

where, when $\varepsilon \downarrow 0$, $\rho(\varepsilon) \rightarrow 0$.

According Lemma 2.4., we have

$$E \int_t^{t+\delta} [|Y_s^{1;v} - Y_s^{2;v}|^2 + |Z_s^{1;v} - Z_s^{2;v}|^2] ds \leq CE \int_t^{t+\delta} \rho^2(d(X_s^{t,x;v}, x)) ds \leq C\delta E\rho^2(\eta^\delta). \quad (3.7)$$

On the other hand, since $Y_t^{1;v}$ and $Y_t^{2;v}$ are both deterministic when $v(\cdot) \in \mathcal{U}_{t,t+\delta}^t$, apply Itô's formula to $Y_s^{1;v} - Y_s^{2;v}$ on $[t, t+\delta]$, we have

$$\begin{aligned} |Y_t^{1;v} - Y_t^{2;v}| &= |E(Y_t^{1;v} - Y_t^{2;v})| \\ &\leq E \int_t^{t+\delta} |F(s, X_s^{t,x;v}, Y_s^{1;v}, Z_s^{1;v}, v_s) - F(s, x, Y_s^{2;v}, Z_s^{2;v}, v_s)| ds \\ &\leq E \int_t^{t+\delta} [K|Y_s^{1;v} - Y_s^{2;v}| + K|Z_s^{1;v} - Z_s^{2;v}| + C\rho(d(X_s^{t,x;v}, x))] ds \\ &\leq C\delta E\rho(\eta^\delta) + C\delta^{\frac{1}{2}} \{E \int_t^{t+\delta} [|Y_s^{1;v} - Y_s^{2;v}|^2 + |Z_s^{1;v} - Z_s^{2;v}|^2] ds\}^{\frac{1}{2}}. \end{aligned}$$

This with (3.7), we get

$$|Y_t^{1;v} - Y_t^{2;v}| \leq C\delta [E\rho(\eta^\delta) + \{E\rho^2(\eta^\delta)\}^{\frac{1}{2}}].$$

Because of the compactness of M , there exists a constant C , such that, $\eta^\delta \leq C, \forall \delta \geq 0$. Thus

$$E\rho^2(\eta^\delta) < \infty.$$

This, together with (3.6) we have that (3.5) holds true.

□

The following lemma tells us how to compute $\inf_{v(\cdot) \in \mathcal{U}_{t,t+\delta}^t} Y_t^{2;v\cdot}$:

Lemma 3.5. We have

$$\inf_{v(\cdot) \in \mathcal{U}_{t,t+\delta}^t} Y_t^{2;v\cdot} = Y_0(t),$$

where $Y_0(t)$ is the solution to the following ODE:

$$\begin{cases} -\dot{Y}_0(s) = F_0(s, x, Y_0(s), 0), s \in [t, t + \delta], \\ Y_0(t + \delta) = 0, \end{cases} \quad (3.8)$$

and the function F_0 is defined as:

$$F_0(t, x, y, z) := \inf_{v \in U} F(t, x, y, z, v).$$

Proof: Consider the following BSDE:

$$\begin{cases} -dY_s^0 = F_0(s, x, Y_s^0, Z_s^0)ds - Z_s^0 dW_s, s \in [t, t + \delta], \\ Y_{t+\delta}^0 = 0. \end{cases} \quad (3.9)$$

Note that F_0 is a deterministic function of (t, x, y, z) . So the solution to (3.9) is just:

$$(Y_s^0, Z_s^0) = (Y_0(s), 0), s \in [t, t + \delta],$$

that is to say equation (3.8) and BSDE (3.9) are the same one.

By the definition of F_0 , we know

$$F_0(s, x, y, z) \leq F(s, x, y, z, v_s), \quad \forall v(\cdot) \in \mathcal{U}_{t,t+\delta}^t, s \in [t, t + \delta].$$

Through the comparison theorem of the solutions to BSDE (3.4) and (3.9), $\forall v(\cdot) \in \mathcal{U}_{t,t+\delta}^t$,

$$Y_s^0 \leq Y_s^{2;v\cdot}, s \in [t, t + \delta].$$

So

$$Y_0(t) = Y_t^0 \leq \inf_{v(\cdot) \in \mathcal{U}_{t,t+\delta}^t} Y_t^{2;v\cdot}.$$

On the other hand, if we denote by $\mathcal{U}_{t,t+\delta}^0$ the set of all admissible controls in $[t, t + \delta]$ which are deterministic processes. Then we can show that

$$Y_t^0 \geq \inf_{v(\cdot) \in \mathcal{U}_{t,t+\delta}^0} Y_t^{2;v\cdot}.$$

That is because, $\forall v(\cdot) \in \mathcal{U}_{t,t+\delta}^0$, $(Y_s^{2;v}, s \in [t, t+\delta])$ is the solution to the following ODE:

$$\begin{cases} -\dot{Y}^{2;v}(s) = F(s, x, Y^{2;v}(s), 0, v_s), s \in [t, t+\delta], \\ Y^{2;v}(t+\delta) = 0. \end{cases}$$

According to the definition of F , for all $\varepsilon > 0$, there exists $(v_s(\varepsilon), s \in [t, t+\delta]) \in \mathcal{U}_{t,t+\delta}^0$, such that,

$$F_0(s, x, y, z) \geq F(s, x, y, z, v_s(\varepsilon)) - \frac{\varepsilon}{\delta}, s \in [t, t+\delta].$$

That yields

$$Y_t^0 \geq Y_t^{2;v(\varepsilon)} - \varepsilon.$$

Since ε is arbitrary, we have

$$Y_t^0 \geq \inf_{v(\cdot) \in \mathcal{U}_{t,t+\delta}^0} Y_t^{2;v}.$$

And consequently

$$Y_0(t) = Y_t^0 \geq \inf_{v(\cdot) \in \mathcal{U}_{t,t+\delta}^0} Y_t^{2;v} \geq \inf_{v(\cdot) \in \mathcal{U}_{t,t+\delta}^t} Y_t^{2;v}.$$

□

After the above preparation, we can show the proof of Theorem 3.2.

The proof of Theorem 3.2. For all $\varphi \in C^{1,2}([0, T \times M])$, suppose that (t, x) is a minimum(resp., maximum) point of $u - \varphi$ and $u(t, x) - \varphi(t, x) = 0$. By DPP(2.8), we have

$$\varphi(t, x) = u(t, x) = \inf_{v(\cdot) \in \mathcal{U}_{t,T}^t} G_{t,t+\delta}^{t,x;v}[u(t+\delta, X_{t+\delta}^{t,x;v})].$$

Since

$$u(t+\delta, X_{t+\delta}^{t,x;v}) \geq \varphi(t+\delta, X_{t+\delta}^{t,x;v}),$$

this with the comparison theorem of the solutions to BSDEs,

$$\inf_{v(\cdot) \in \mathcal{U}_{t,T}^t} \{G_{t,t+\delta}^{t,x;v}[\varphi(t+\delta, X_{t+\delta}^{t,x;v})] - \varphi(t, x)\} \leq 0 \quad (\text{resp., } \geq 0).$$

Through Lemma 3.3., we have

$$\inf_{v(\cdot) \in \mathcal{U}_{t,T}^t} Y_t^{1;v} \leq 0 \quad (\text{resp., } \geq 0).$$

So by (3.5),

$$\inf_{v(\cdot) \in \mathcal{U}_{t,T}^t} Y_t^{2;v} \leq C\delta\rho_1(\delta) \quad (\text{resp., } \geq -C\delta\rho_1(\delta)).$$

Accordinging Lemma 3.5., it yields

$$Y_0(t) \leq C\delta\rho_1(\delta) \quad (\text{resp., } \geq -C\delta\rho_1(\delta)).$$

So

$$\begin{aligned} Y_0(t) &= \int_t^{t+\delta} F_0(s, x, Y_0(s), 0) ds \\ &= \delta F_0(t + \delta, x, Y_0(t + \delta), 0) + o(\delta) \\ &= \delta F_0(t + \delta, x, 0, 0) + o(\delta) \\ &\leq C\delta\rho_1(\delta) \quad (\text{resp., } \geq -C\delta\rho_1(\delta)). \end{aligned}$$

Divided by δ and let $\delta \downarrow 0$, we have

$$F_0(t, x, 0, 0) = \inf_{v \in U} F(t, x, 0, 0, v) \leq 0 \quad (\text{resp., } \geq 0).$$

That is to say

$$\partial_t \varphi(t, x) + \inf_{v \in U} \{ (v_0 V_0 \varphi)(t, x) + \frac{1}{2} \sum_{\alpha=1}^d (v_\alpha^2 V_\alpha V_\alpha \varphi)(t, x) + f(t, x, u, \{v_\alpha V_\alpha \varphi\}_{\alpha=1}^d, v) \} \leq 0, \quad (\text{resp., } \geq 0).$$

And obviously, $u(T, x) = \Phi(x)$. So $u(t, x)$ is a viscosity solution to PDE (3.1).

Now let's deal with the uniqueness conclusion. We assume that for all $x, y \in M$ s.t. $d(x, y) < \min\{i_M(x), i_M(y)\}$, $t \in [0, T]$,

$$\begin{aligned} (H1) \quad & \|L_{xy}V_0(t, x) - V_0(t, y)\| \leq \mu d(x, y), \\ (H2) \quad & L_{xy}V_\alpha(t, x) = V_\alpha(t, y), \alpha = 1, 2, \dots, d, \end{aligned}$$

where μ is a positive constant.

Consider a generalized case:

$$\begin{cases} u_t + \inf_{v \in U} H(t, x, u, du, d^2u, v) = 0 & \text{in } (0, T) \times M, \\ u(0, x) = \psi(x), x \in M, \end{cases} \quad (3.10)$$

where, du, d^2u mean $d_x u(t, x)$ and $d_x^2 u(t, x)$.

Set

$$\chi := \{(t, x, r, \zeta, A, v) : t \in [0, T], x \in M, r \in R, \zeta \in TM_x^*, A \in \mathcal{L}_s^2(TM_x), v \in U\},$$

where TM_x^* stands for the cotangent space of M at a point x , TM_x stands for the tangent space at x and $\mathcal{L}_s^2(TM_x)$ denotes the symmetric bilinear forms on TM_x .

From [7], we have the following comparison theorem of viscosity solutions to PDE (3.10):

Theorem 3.6. Let M be a compact Riemannian manifold (without boundary), and $H : \chi \rightarrow R$ be continuous, proper for each fixed $(t, v) \in (0, T) \times U$ and

satisfy: there exists a function $\omega : [0, +\infty] \rightarrow [0, +\infty]$ with $\omega(0+) = 0$ and such that

$$\sup_{v \in U} [H(t, y, r, \alpha \exp_y^{-1}(x), Q, v) - H(t, x, r, -\alpha \exp_x^{-1}(y), P, v)] \leq \omega(\alpha d^2(x, y) + d(x, y)), \quad (3.11)$$

for all fixed $t \in (0, T)$ and for all $x, y \in M, r \in R, P \in T_{2,s}(M)_x, Q \in T_{2,s}(M)_y$ with

$$-(\frac{1}{\varepsilon_\alpha} + \|A_\alpha\|) \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} P & 0 \\ 0 & -Q \end{pmatrix} \leq A_\alpha + \varepsilon_\alpha A_\alpha^2, \quad (3.12)$$

where A_α is the second derivative of the function $\varphi_\alpha(x, y) = \frac{\alpha}{2}d^2(x, y)$ ($\alpha > 0$) at the point $(x, y) \in M \times M$,

$$\varepsilon_\alpha = \frac{1}{2(1 + \|A_\alpha\|)}$$

and the points x, y are assumed to be close enough to each other so that $d(x, y) < \min\{i_M(x), i_M(y)\}$.

Let $u \in USC([0, T] \times M)$ be a subsolution and $v \in LSC([0, T] \times M)$ a supersolution of (3.10). Then $u \leq v$ on $[0, T] \times M$. In particular PDEs (3.10) has at most one viscosity solution.

Theorem 3.7. The value function $u(t, x)$ is the unique viscosity solution to PDE(3.1).

Proof: For PDE(3.1), the function $H : [0, T] \times M \times R \times TM_x^* \times \mathcal{L}_s^2(TM_x) \times U \rightarrow R$ is:

$$\begin{aligned} & H(t, x, r, \zeta, P, v) \\ &= -f(t, x, r, \langle \zeta, v_\alpha V_\alpha(t, x) \rangle_{\alpha=1}^d, v) - \langle \zeta, v_0 V_0(t, x) \rangle - \frac{1}{2} \sum_{\alpha=1}^d v_\alpha^2 \langle P V_\alpha(t, x), V_\alpha(t, x) \rangle. \end{aligned}$$

So for any fixed $t \in (0, T), r \in R$, when $d(x, y) < \frac{1}{2}i_M$, if $P \in \mathcal{L}_s^2(TM_x), Q \in \mathcal{L}_s^2(TM_y)$ satisfy the following matrix inequality:

$$-(\frac{1}{\varepsilon_\alpha} + \|A_\alpha\|) \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} P & 0 \\ 0 & -Q \end{pmatrix} \leq A_\alpha + \varepsilon_\alpha A_\alpha^2,$$

where A_α is the second derivative of the function $\varphi_\alpha(x, y) = \frac{\alpha}{2}d^2(x, y)$ at the point $(x, y) \in M \times M$,

$$\varepsilon_\alpha = \frac{1}{2(1 + \|A_\alpha\|)}.$$

Since M is compact, there is a $k_0 > 0$ s.t. the sectional curvature is bounded below by $-k_0$ on M . So by Remark 4.7 in [1],

$$P - L_{yx}(Q) \leq \frac{3}{2}k_0\alpha d^2(x, y)I.$$

Then

$$\begin{aligned}
& \sup_{v \in U, t \in (0, T)} [H(t, y, r, \alpha \exp_y^{-1}(x), Q, v) - H(t, x, r, -\alpha \exp_x^{-1}(y), P, v)] \\
= & \sup_{v \in U, t \in (0, T)} \{f(t, x, r, \langle -\alpha \exp_x^{-1}(y), v_\alpha V_\alpha(t, x) \rangle_{\alpha=1}^d, v) \\
& - f(t, y, r, \langle \alpha \exp_y^{-1}(x), v_\alpha V_\alpha(t, y) \rangle_{\alpha=1}^d, v) \\
& + \frac{1}{2} \sum_{\alpha=1}^d v_\alpha^2 [\langle P V_\alpha(t, x), V_\alpha(t, x) \rangle - \langle Q V_\alpha(t, y), V_\alpha(t, y) \rangle] \\
& - \langle \alpha \exp_x^{-1}(y), v_0 V_0(t, x) \rangle - \langle \alpha \exp_y^{-1}(x), v_0 V_0(t, y) \rangle\} \\
\leq & \sup_{v \in U, t \in (0, T)} \{Kd(x, y) + \frac{3}{4} k_0 \alpha d^2(x, y) \sum_{\alpha=1}^d v_\alpha^2 \langle V_\alpha, V_\alpha \rangle_{(t, x)} \\
& + \langle L_{yx} \alpha \exp_y^{-1}(x), v_0 V_0(t, x) \rangle - \langle \alpha \exp_y^{-1}(x), v_0 V_0(t, y) \rangle\} \\
\leq & \bar{C}(\alpha d^2(x, y) + d(x, y)).
\end{aligned}$$

Since M and U are both compact, through (A1), (H1) and (H2), we get the last inequality. Where \bar{C} is a constant positive. By Theorem 3.6. we can get the uniqueness result of the viscosity solution to PDE (3.1).

□

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